Tangent plane revisited

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Not all surfaces can be described as a explicit function, $z = f(x, y)$. Usually we are not able to isolate z in a surface equation, e.g., $x + y + ze^z = 0$. Even a surface as simple as the sphere is not the graph of any single function of two variables.
But, of course, a sphere is a smooth surface and it must have a tangent plane at any point.
In general, we have an equation of the form $F(x, y, z) = c$. We wonder how to find the tangent planes of a surface on that implicit form. Let's start with a particular case the functions, $z = f(x, y)$:

$$\begin{aligned} z &= f(x,y) \\ z - f(x,y) &= 0 \\ F(x,y,z) := z - f(x,y) &= 0 \\ \text{So, } \nabla F(x,y,z) &= (-f_x, -f_y, 1) \end{aligned}$$

On the other hand:

$$\begin{aligned} \pi: \quad z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ \pi: \quad z - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b) = 0 \\ \pi: \quad (-f_x, -f_y, 1) \cdot \Big(x-a, y-b, z-f(a,b)\Big) = 0 \end{aligned}$$

So we may suspect that in general, the tangent plane equation is:

$$\pi: \nabla F(x,y,z) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Effectively, we have the following result:

Let S be an hypersurface in $\mathbb{R}^n$ defined by an equation of the form $F(\mathbf{x}) = c,\ \mathbf{x} \in \mathbb{R}^n$.
If $F(x, y, z)$ is a differentiable function (or class C^1 then the tangent hyperplane at a point x_0 on the hypersurface S is

$$\nabla F(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x_0}) = 0, \text{ if } \nabla F(\mathbf{x}) \text{ is nonzero}$$

For $n = 3$ we have (tangent plane implicit form):

$$\begin{aligned} \nabla F(x_0,y_0,z_0) \cdot \Big[(x,y,z) - (x_0,y_0,z_0)\Big] = 0 \\ F_x \cdot (x - x_0) + F_y \cdot (y - y_0) + F_z \cdot (z - z_0) = 0 \end{aligned}$$

For $n = 2$ we have (tangent line implicit form):

$$\nabla F(x_0, y_0) \cdot \Big[(x, y) - (x_0, y_0)\Big] = 0$$
$$F_x(x - x_0) + F_y(y - y_0) = 0$$

Observation: $\nabla F(x_0, y_0, z_0) \perp S_{x_0}$.

Example:

Find the tangent line of the surface $x^2 + y^2 = 1$ at the point $(0, 1)$.

$$F(x, y) = x^2 + y^2 - 1$$
$$\nabla F(x, y) = (2x, 2y)$$
$$\nabla F(0, 1) = (0, 2)$$

So,

$$l : (0, 2)(x - 0, y - 1) = 0$$
$$l : 2y - 2 = 0$$
$$l : y = 1$$

Example:

Consider the hyperboloid: $3x^2 - 9y^2 + z^2 = 10$. Find the points where the tangent plane is parallel to the plane $\pi : -6x + 18y + 8z = 7$.

Let's consider the implicit function $F(x, y, z) = 3x^2 - 9y^2 + z^2$. Then $\nabla F(x, y, z)$ must be perpendicular to the tangent plane at (x, y, z). That's is $\nabla F(x, y, z)$ must be on the direction of $(-6, 18, 8)$:

$$\nabla F(x, y, z) = k(-6, 18, 8), k \in \mathbb{R}$$
$$(6x, -18y, 2z) = k(-6, 18, 8), \text{ so}$$
$$6x = -6k \longrightarrow x = -k$$
$$-18y = 18k \longrightarrow y = -k$$
$$2z = 8k \longrightarrow z = 4k$$

The point/s must also satisfy the equation of the surface, i.e.:

$$F(-k, -k, 4k) = 10$$
$$3(-k)^2 - 9(-k)^2 + (4k)^2 = 10$$
$$10k^2 = 10$$
$$k = \pm 1$$

Then the points are: $(-1,-1,4),(1,1,-k)$.

Example:

This problem concerns the surface defined by the equation

$$x^3z + x^2y^2 + \sin yz = -3$$

1. Find an equation for the plane tangent to this surface at the point (-1, 0, 3).
2. The **normal** line to a surface S in $\mathbb{R}^3$ at a point (x_0, y_0, z_0) on it is the line that passes through (x_0, y_0, z_0) and is perpendicular to S. Find a set of parametric equations for the line normal to the surface given above at the point $(-1, 0, 3)$.

1.

Let $F(x,y,z) = x^3z + x^2y^2 + \sin yz$. Then

$$\nabla F(x,y,z) = (3x^2 + 2xy^2, 2x^2y + z\cos yz, x^3 + y\cos yz))$$

So,

$$\begin{aligned} \pi : \nabla F(-1,0,3) \cdot (x+1, y, z-3) = 0 \\ \pi : 9(x+1) + 3y - (z-3) = 0 \\ \pi : 9x + 3y - z = -12 \end{aligned}$$

2.

The director vector of the normal line is the has the same direction as the gradient since $\nabla F \perp S$. So,

$$n(t) : \begin{cases} x = 9t - 1 \\ y = 3t \\ z = -t + 3 \end{cases}$$